# Invariance of the geometrical phase under time-dependent unitary transformations



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# Invariance of the geometrical phase under time-dependent unitary transformations

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**Abstract.** The effects of time-dependent unitary transformations on the geometrical phase are investigated in the general non-adiabatic setting. A detailed study considering both paths of an interference experiment shows that even though the geometry of the fibre bundle: Hilbert space → space of states changes. The measured relative phase, geometrical phase and dynamical phase are all invariant under these transformations.

## 1. Introduction

The effects of time-dependent unitary transformations on the geometric (Berry) phase have been discussed by several authors in different contexts. In the adiabatic treatments [1–3] the Berry phase was said to become 'dynamical' under these transformations. By becoming 'dynamical' it loses geometrical meaning and could in this way be 'removed' [1, 2]. It was then shown that even though the Berry phase could become 'dynamical', it retained its geometrical meaning and was therefore not 'removed' [3]. In non-adiabatic treatments of specific physical systems [4, 5] the Berry phase has been recently generalized so that it was invariant under time-dependent unitary transformations. A distinction between the Hamiltonian and energy operator was made in order to define the dynamical phase in an invariant way. A general non-adiabatic treatment was briefly discussed for periodic transformations in terms of relative frames [6].

We will consider arbitrary time-dependent unitary transformations and work in the general non-adiabatic setting. We will begin by reviewing a simple interference experiment in order to clearly define the measured relative phase. This phase is expressed in terms of the usual dynamical and geometrical parts. The effects of time-dependent unitary transformations will then be investigated in detail. These transformations alter the time evolution equations for both paths in an interference experiment leaving the measured relative phase invariant. By carefully considering the time evolution of both paths, we will show that the geometrical and dynamical phases are also invariant, and that the geometrical phase remains entirely geometrical. We will see explicitly what part of the phase should be called dynamical, and relate these ideas to the above-mentioned energy operator.

## 2. The interference experiment

The measurement of a relative phase is accomplished by performing an interference experiment. This experiment measures the relative phase between two state vectors which have undergone different time evolutions, but represent the same initial and final physical state. A standard example is the following [7]. Consider a beam of neutral particles whose magnetic moment and spin axis coincide. Split the beam into two paths, one path going through an apparatus which contains a magnetic field and the other path going around the apparatus in free space. The magnetic field may or may not be time varying [8], and is chosen so that when the beam exits the apparatus it is in the same quantum state as when it entered. Recombine the beams and observe the interference pattern. If the transit times of the two paths are set equal, the dynamical phase produced by the free space Hamiltonian for each path will be equal. This common phase will then cancel when the beams are recombined. The remaining relative phase between the two paths represents the effect of the apparatus which is not common to both paths. Although this particular example is not practical, it is a simple theoretical model which can be used to describe interference experiments.

In order to derive the measured relative phase, we must consider the time evolution equations for both paths. We will denote path 1 as the path which goes through the apparatus and path 2 as the path which goes around the apparatus. We will assume that the transit times have been set equal so that we can ignore the free space Hamiltonian. The time evolution equations for both paths are

$$\mathrm{i}|\dot{\psi}_1(t)\rangle = h(t)|\psi_1(t)\rangle \tag{1}$$

$$\mathrm{i}|\dot{\psi}_2(t)\rangle = 0 \tag{2}$$

where $|\psi_1(t)\rangle$ is the state vector representing the state of path 1, $|\psi_2(t)\rangle$ is the state vector representing the state of path 2 and $h(t)$ is the Hamiltonian representing the apparatus. We will consider a time-dependent system for which the Hamiltonian $h(t)$ is equal to the energy of the system. However, for a time-dependent system the energy is in general different from the Hamiltonian [4, 5]. The Hamiltonian $h(t)$ is chosen so as to produce cyclic evolution for path 1. By cyclic evolution we mean that the curve in the space of physical states (projective Hilbert space) is closed. We will denote the physical states of path 1 and 2 by the projection operators

$$\pi_1(t) \equiv |\psi_1(t)\rangle\langle\psi_1(t)| \tag{3a}$$

$$\pi_2(t) \equiv |\psi_2(t)\rangle\langle\psi_2(t)|. \tag{3b}$$

We can express the condition for cyclic evolution over a time period $T$ as

$$\pi_1(T) = \pi_1(0). \tag{4}$$

At time $t = 0$ (when the beam is split) the states of the two paths are set equal

$$\pi_1(0) = \pi_2(0). \tag{5}$$

From equation (2) we see that the time evolution of the state vector $|\psi_2(t)\rangle$ is given by

$$|\psi_2(t)\rangle = |\psi_2(0)\rangle \qquad \forall t \tag{6}$$

which implies that the state of path 2 is stationary

$$\pi_2(t) = \pi_2(0) \qquad \forall t. \tag{7}$$

From equations (4), (5) and (7) we see that the final states of path 1 and 2 are equal

$$\pi_1(T) = \pi_2(T). \tag{8}$$

The measured relative phase between the two state vectors (which represent the same physical state at time T) is given by

$$|\psi_1(T)\rangle = e^{i\alpha}|\psi_2(T)\rangle. \tag{9}$$

Let us define the phase change of path 1 and path 2 by

$$|\psi_1(T)\rangle = e^{i\alpha_1}|\psi_1(0)\rangle \tag{10}$$

$$|\psi_2(T)\rangle = e^{i\alpha_2}|\psi_2(0)\rangle. \tag{11}$$

We can choose without loss of generality

$$|\psi_1(0)\rangle = |\psi_2(0)\rangle \tag{12}$$

and by substituting equation (12) into equation (11) and using this result in equation (10) we find

$$|\psi_1(T)\rangle = e^{i(\alpha_1-\alpha_2)}|\psi_2(T)\rangle. \tag{13}$$

By comparing equation (13) to equation (9) we see

$$e^{i\alpha} = e^{i(\alpha_1-\alpha_2)}$$

or

$$\alpha = \alpha_1 - \alpha_2 \qquad (\mathrm{mod}\ 2\ \pi). \tag{14}$$

From equation (6) we see that $\alpha_2 = 0$ and from equation (14) this implies that $\alpha = \alpha_1$. In the usual way we can express the total relative phase $\alpha$ in terms of a geometrical and dynamical part [8]. The connection defines this splitting [9, 10]

$$\alpha = \alpha_1 = i\int_0^T \langle\phi(t)|\dot{\phi}(t)\rangle\,dt - \int_0^T \langle\psi_1(t)|h(t)|\psi_1(t)\rangle\,dt \tag{15}$$

where the vectors $|\phi(t)\rangle$ are related to the vectors $|\psi_1(t)\rangle$ by a phase such that $|\phi(T)\rangle = |\phi(0)\rangle$. The geometric phase is

$$\beta = i\int_0^T \langle\phi(t)|\dot{\phi}(t)\rangle\,dt. \tag{16}$$

By introducing coordinates $x$ of projective Hilbert space, the geometric phase $\beta$ can be written in terms of the connection form $A$ defined by

$$A \equiv i\langle\phi(x)|d|\phi(x)\rangle \tag{17}$$

where $d$ is the exterior derivative with respect to the coordinates $x$. Equation (16) becomes

$$\beta = \oint_c A \tag{18}$$

where $c$ is the closed curve in projective Hilbert space. The dynamical phase is

$$\delta = -\int_0^T \langle\psi_1(t)|h(t)|\psi_1(t)\rangle\,dt. \tag{19}$$

## 3. Time dependent unitary transformations

We want to consider time-dependent unitary transformations acting on Hilbert space. All the vectors transform according to

$$|\tilde{\psi}(t)\rangle = U(t)|\psi(t)\rangle \tag{20}$$

where

$$U(t) \cdot U^{\dagger}(t) = 1 \qquad \forall t.$$

It is important to note that the unitary transformation given by $U(t) = \mathrm{e}^{\mathrm{i}f(t)}$ (where $f(t)$ is a real function) is not a gauge transformation. By definition a gauge transformation corresponds to a change in the local section which occurs upon choosing a new coordinate patch in projective Hilbert space [9]. A local section is a continuous mapping of a patch in projective Hilbert space into Hilbert space itself. A closed curve in projective Hilbert space is mapped by a local section into a closed curve in Hilbert space. We will denote the closed curve in Hilbert space by $|\phi(t)\rangle$. We can express a state vector in terms of $|\phi(t)\rangle$ by

$$|\psi(t)\rangle = \mathrm{e}^{\mathrm{i}\xi(t)}|\phi(t)\rangle \tag{21}$$

where

$$|\phi(T)\rangle = |\phi(0)\rangle. \tag{22}$$

A gauge transformation is given by

$$|\phi'(t)\rangle = \mathrm{e}^{\mathrm{i}\theta(t)}|\phi(t)\rangle \tag{23}$$

where

$$\theta(T) = \theta(0) + 2\pi n \qquad (n \text{ an integer}).$$

Substituting equation (23) into equation (21) we find

$$|\psi(t)\rangle = \mathrm{e}^{\mathrm{i}\xi(t)}\mathrm{e}^{-\mathrm{i}\theta(t)}|\phi'(t)\rangle$$

which we can write as

$$|\psi(t)\rangle = \mathrm{e}^{\mathrm{i}\xi'(t)}|\phi'(t)\rangle \tag{24}$$

where

$$\mathrm{e}^{\mathrm{i}\xi'(t)} = \mathrm{e}^{\mathrm{i}\xi(t)}\mathrm{e}^{-\mathrm{i}\theta(t)}.$$

By comparing equations (20), (21) and (24) we see that a unitary transformation changes the vector $|\psi(t)\rangle$ whereas a gauge transformation changes $|\phi(t)\rangle$.

Writing equation (20) in terms of $|\psi(t)\rangle$

$$|\psi(t)\rangle = U^{\dagger}(t)|\tilde{\psi}(t)\rangle$$

and substituting this expression into the Schrödinger equation for $|\psi(t)\rangle$

$$\mathrm{i}|\dot{\psi}(t)\rangle = h(t)|\psi(t)\rangle$$

we find the effective Schrödinger equation for $|\tilde{\psi}(t)\rangle$

$$\mathrm{i}|\dot{\tilde{\psi}}(t)\rangle = \tilde{h}(t)|\tilde{\psi}(t)\rangle \tag{25}$$

where

$$\tilde{h}(t) = U(t)h(t)U^{\dagger}(t) + \mathrm{i}\dot{U}(t)U^{\dagger}(t). \tag{26}$$

The transformation $U(t)$ is acting on all the vectors in Hilbert space. It therefore effects the time evolution equations of *both* path 1 and path 2 of our interference experiment. Equations (1) and (2) become

$$\mathrm{i}|\dot{\tilde{\psi}}_1(t)\rangle = \tilde{h}_1(t)|\tilde{\psi}_1(t)\rangle \tag{27$a$}$$

$$\mathrm{i}|\dot{\tilde{\psi}}_2(t)\rangle = \tilde{h}_2(t)|\tilde{\psi}_2(t)\rangle \tag{27$b$}$$

where

$$\tilde{h}_1(t) = U(t)h(t)U^{\dagger}(t) + \mathrm{i}\dot{U}(t)U^{\dagger}(t) \tag{28$a$}$$

$$\tilde{h}_2(t) = 0 + \mathrm{i}\dot{U}(t)U^{\dagger}(t). \tag{28$b$}$$

Under the transformation $U(t)$ the curves in projective Hilbert space represented by the projection operators $\pi_1(t)$ and $\pi_2(t)$ change according to

$$\tilde{\pi}_1(t) = U(t)\pi_1(t)U^{\dagger}(t) \tag{29$a$}$$

$$\tilde{\pi}_2(t) = U(t)\pi_2(t)U^{\dagger}(t). \tag{29$b$}$$

Before proceeding we will show that the measured relative phase $\alpha$ is invariant under the transformation $U(t)$. The transformed vectors at time $T$ can be expressed as

$$\begin{aligned}|\tilde{\psi}_1(T)\rangle &= U(T)|\psi_1(T)\rangle \\ &= U(T)\mathrm{e}^{\mathrm{i}\alpha_1}|\psi_1(0)\rangle \end{aligned} \tag{30a}$$

$$\begin{aligned}|\tilde{\psi}_2(T)\rangle &= U(T)|\psi_2(T)\rangle \\ &= U(T)\mathrm{e}^{\mathrm{i}\alpha_2}|\psi_2(0)\rangle \end{aligned} \tag{30b}$$

where we have used equations (10) and (11). Using equation (12) in equation (30$b$) and substituting this result into equation (30$a$) we find

$$|\tilde{\psi}_1(T)\rangle = \mathrm{e}^{\mathrm{i}\alpha_1}\mathrm{e}^{-\mathrm{i}\alpha_2}U(T)U^{\dagger}(T)|\tilde{\psi}_2(T)\rangle$$

$$|\tilde{\psi}_1(T)\rangle = \mathrm{e}^{\mathrm{i}(\alpha_1-\alpha_2)}|\tilde{\psi}_2(T)\rangle$$

and using equation (14)

$$|\tilde{\psi}_1(T)\rangle = \mathrm{e}^{\mathrm{i}\alpha}|\tilde{\psi}_2(T)\rangle. \tag{31}$$

Comparing this result with equation (9) we see that the measured relative phase in an interference experiment is invariant under general time-dependent unitary transformations.

For future reference we note that at $t = 0$

$$|\tilde{\psi}_1(0)\rangle = U(0)|\psi_1(0)\rangle \tag{32$a$}$$

$$|\tilde{\psi}_2(0)\rangle = U(0)|\psi_2(0)\rangle \tag{32$b$}$$

and upon substituting equation (12) into equation (32$b$) and using this result in equation (32$a$) we find

$$|\tilde{\psi}_1(0)\rangle = |\tilde{\psi}_2(0)\rangle. \tag{33}$$

We will begin our analysis with the simple case

$$U(t) = \mathrm{e}^{\mathrm{i}f(t)}. \tag{34}$$

From equations (29$a$) and (29$b$) we see that for $U(t)$ given by equation (34) the paths $\pi_1(t)$ and $\pi_2(t)$ are unchanged. The time evolution equations (1) and (2) become according to equations (27) and (28)

$$i|\dot{\tilde{\psi}}_1(t)\rangle = \left[h(t) - \dot{f}(t)\right]|\tilde{\psi}_1(t)\rangle \tag{35$a$}$$

$$i|\dot{\tilde{\psi}}_2(t)\rangle = -\dot{f}(t)|\tilde{\psi}_2(t)\rangle. \tag{35$b$}$$

Let us define the phase $\tilde{\alpha}_1$ for the transformed path 1 by

$$|\tilde{\psi}_1(T)\rangle = \mathrm{e}^{\mathrm{i}\tilde{\alpha}_1}|\tilde{\psi}_1(0)\rangle. \tag{36}$$

Since the curve in projective Hilbert space is unchanged, we can express $|\tilde{\psi}_1(t)\rangle$ in terms of $|\phi(t)\rangle$ (see equations (21) and (22))

$$|\tilde{\psi}_1(t)\rangle = \mathrm{e}^{\mathrm{i}\tilde{\xi}_1(t)}|\phi(t)\rangle \tag{37$a$}$$

where

$$\tilde{\xi}_1(T) - \tilde{\xi}_1(0) = \tilde{\alpha}_1. \tag{37$b$}$$

Substituting equation (37$a$) into equation (35$a$), contracting on the left with $\langle\tilde{\psi}_1|$, integrating and using equation (37$b$) we find

$$\tilde{\alpha}_1 = \mathrm{i}\int_0^T \langle\phi(t)|\dot{\phi}(t)\rangle\,\mathrm{d}t - \int_0^T \langle\psi_1(t)|h(t)|\psi_1(t)\rangle\,\mathrm{d}t + \int_0^T \dot{f}(t)\,\mathrm{d}t. \tag{38}$$

At this point we must remember that the measured phase is $\alpha$ not $\tilde{\alpha}_1$. We must do a similar analysis for path 2 where we define the phase $\tilde{\alpha}_2$ by

$$|\tilde{\psi}_2(T)\rangle = \mathrm{e}^{\mathrm{i}\tilde{\alpha}_2}|\tilde{\psi}_2(0)\rangle. \tag{39}$$

Equation (35$b$) is easily integrated to find

$$\tilde{\alpha}_2 = \int_0^T \dot{f}(t)\,\mathrm{d}t. \tag{40}$$

Using equation (33) in equation (39) and then substituting this result into equation (36) we find (using also equation (31))

$$\alpha = \tilde{\alpha}_1 - \tilde{\alpha}_2 = \mathrm{i}\int_0^T \langle\phi(t)|\dot{\phi}(t)\rangle\,\mathrm{d}t - \int_0^T \langle\psi_1(t)|h(t)|\psi_1(t)\rangle\,\mathrm{d}t \tag{41}$$

which we see is the original expression equation (15). By carefully considering the time evolution of both paths in our interference experiment, we have seen explicitly that the total measured phase, geometrical phase and dynamical phase are all invariant under the transformation given by equation (34). This is easily understood from the fact that the curve in projective Hilbert space has not changed.

Now let us consider general time-dependent unitary transformations which can change the curve in projective Hilbert space. We have already shown that the measured relative phase is invariant (see equation (31)). We need to investigate how the splitting of the measured phase into geometrical and dynamical parts is affected. From equation (29$a$) we see that the curve in projective Hilbert space generated by path 1 evolution in our interference experiment is no longer closed. From equation (29$b$) we see that the stationary state (see equation (7)) corresponding to path 2 is no longer stationary. Under the transformation $U(t)$ the time evolution of path 1 and path 2 generate two different curves in projective Hilbert space. The two curves start at the same point at $t = 0$ and later meet at some other point at $t = T$. This can be seen explicitly from equations (29$a$) and (29$b$) upon setting $t = 0$ and $t = T$ and using equations (5) and (8)

$$\tilde{\pi}_1(0) = \tilde{\pi}_2(0) \tag{42a}$$
$$\tilde{\pi}_1(T) = \tilde{\pi}_2(T). \tag{42b}$$

Taken together the two curves form a closed curve in projective Hilbert space. We will now evaluate the measured relative phase using the transformed vectors. We can express $|\tilde{\psi}_1(t)\rangle$ in terms of a vector denoted by $|\phi_1(t)\rangle$

$$|\tilde{\psi}_1(t)\rangle = \mathrm{e}^{\mathrm{i}\lambda_1(t)}|\phi_1(t)\rangle. \tag{43}$$

Substituting equation (43) into equation (27$a$), contracting on the left with $\langle\tilde{\psi}_1(t)|$ and integrating we find

$$\lambda_1(T) - \lambda_1(0) = \mathrm{i}\int_0^T \langle\phi_1(t)|\dot{\phi}_1(t)\rangle\,\mathrm{d}t - \int_0^T \langle\tilde{\psi}_1(t)|\tilde{h}_1(t)|\tilde{\psi}_1(t)\rangle\,\mathrm{d}t. \tag{44}$$

Doing a similar analysis for $|\tilde{\psi}_2(t)\rangle$ we write

$$|\tilde{\psi}_2(t)\rangle = \mathrm{e}^{\mathrm{i}\lambda_2(t)}|\phi_2(t)\rangle. \tag{45}$$

We want to combine $|\phi_1(t)\rangle$ and $|\phi_2(t)\rangle$ so that they form a closed curve in Hilbert space (analogous to $|\phi(t)\rangle$). In order to accomplish this, we choose $|\phi_1(t)\rangle$ and $|\phi_2(t)\rangle$ so that they meet at their endpoints

$$|\phi_2(0)\rangle = |\phi_1(0)\rangle \tag{46a}$$
$$|\phi_2(T)\rangle = |\phi_1(T)\rangle. \tag{46b}$$

Substituting equation (45) into equation (27$b$) and performing the same steps that led to equation (44) we find

$$\lambda_2(T) - \lambda_2(0) = \mathrm{i}\int_0^T \langle\phi_2(t)|\dot{\phi}_2(t)\rangle\,\mathrm{d}t - \int_0^T \langle\tilde{\psi}_2(t)|\tilde{h}_2(t)|\tilde{\psi}_2(t)\rangle\,\mathrm{d}t. \tag{47}$$

Setting $t = 0$ in equations (43) and (45) and using equation (33) and (46$a$) we find

$$e^{i\lambda_2(0)} = e^{i\lambda_1(0)}. \tag{48}$$

Setting $t = T$ in equations (43) and (45) and using equation (31) and (46$b$) we find

$$e^{i\alpha} = e^{i(\lambda_1(T)-\lambda_2(T))}. \tag{49}$$

We subtract equation (47) from equation (44) and use equation (48) and (49) to find (mod $2\pi$)

$$\begin{aligned}\alpha = {} & i\int_0^T \langle\phi_1(t)|\dot{\phi}_1(t)\rangle\,dt - i\int_0^T \langle\phi_2(t)|\dot{\phi}_2(t)\rangle\,dt \\ & - \int_0^T \langle\tilde{\psi}_1(t)|\tilde{h}_1(t)|\tilde{\psi}_1(t)\rangle\,dt + \int_0^T \langle\tilde{\psi}_2(t)|\tilde{h}_2(t)|\tilde{\psi}_2(t)\rangle\,dt.\end{aligned} \tag{50}$$

By introducing the coordinates $x$ of projective Hilbert space, we can express the first two terms on the RHS of equation (50) as line integrals of the connection form $A$ (see equation (17))

$$i\int_0^T \langle\phi_1(t)|\dot{\phi}_1(t)\rangle\,dt = \int_{\tilde{c}_1} A \tag{51$a$}$$

$$-i\int_0^T \langle\phi_2(t)|\dot{\phi}_2(t)\rangle\,dt = -\int_{\tilde{c}_2} A = \int_{-\tilde{c}_2} A \tag{51$b$}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are the two curves in projective Hilbert space represented by the operators $\tilde{\pi}_1(t)$ and $\tilde{\pi}_2(t)$. The closed curve in projective Hilbert space is denoted by

$$\tilde{c} \equiv -\tilde{c}_2 + \tilde{c}_1. \tag{52}$$

By adding equations (51$a$) and (51$b$) we find

$$i\int_0^T \langle\phi_1(t)|\dot{\phi}_1(t)\rangle\,dt - i\int_0^T \langle\phi_2(t)|\dot{\phi}_2(t)\rangle\,dt = \int_{-\tilde{c}_2+\tilde{c}_1} A = \oint_{\tilde{c}} A. \tag{53}$$

Equation (50) can now be expressed as

$$\alpha = \oint_{\tilde{c}} A - \int_0^T \langle\tilde{\psi}_1(t)|\tilde{h}_1(t)|\tilde{\psi}_1(t)\rangle\,dt + \int_0^T \langle\tilde{\psi}_2(t)|\tilde{h}_2(t)|\tilde{\psi}_2(t)\rangle\,dt. \tag{54}$$

We now use equations (28$a$) and (28$b$) in equation (54)

$$\begin{aligned}\alpha = {} & \oint_{\tilde{c}} A - i\int_0^T \langle\tilde{\psi}_1(t)|\dot{U}(t)U^\dagger(t)|\tilde{\psi}_1(t)\rangle\,dt + i\int_0^T \langle\tilde{\psi}_2(t)|\dot{U}(t)U^\dagger(t)|\tilde{\psi}_2(t)\rangle\,dt \\ & - \int_0^T \langle\tilde{\psi}_1(t)|U(t)h(t)U^\dagger(t)|\tilde{\psi}_1(t)\rangle\,dt.\end{aligned} \tag{55}$$

We can express the second and third terms on the RHS of equation (55) as line integrals

$$
\begin{aligned}
-\mathrm{i}\int_0^T \langle\tilde{\psi}_1(t)|\dot{U}(t)U^\dagger(t)|\tilde{\psi}_1(t)\rangle\,\mathrm{d}t &= -\mathrm{i}\int_0^T \langle\phi_1(t)|\dot{U}(t)U^\dagger(t)|\phi_1(t)\rangle\,\mathrm{d}t \\
&= -\mathrm{i}\int_{\tilde{c}_1} \langle\phi(x)|\mathrm{d}U(x)U^\dagger(x)|\phi(x)\rangle
\end{aligned} \tag{56a}
$$

$$
\begin{aligned}
\mathrm{i}\int_0^T \langle\tilde{\psi}_2(t)|\dot{U}(t)U^\dagger(t)|\tilde{\psi}_2(t)\rangle\,\mathrm{d}t &= \mathrm{i}\int_0^T \langle\phi_2(t)|\dot{U}(t)U^\dagger(t)|\phi_2(t)\rangle\,\mathrm{d}t \\
&= \mathrm{i}\int_{\tilde{c}_2} \langle\phi(x)|\mathrm{d}U(x)U^\dagger(x)|\phi(x)\rangle .
\end{aligned} \tag{56b}
$$

where we have used equations (43) and (45) and expressed all quantities in terms of the coordinates $x$ of projective Hilbert space. We now add equations (56$a$) and (56$b$) to obtain

$$
-\mathrm{i}\int_{\tilde{c}_1-\tilde{c}_2} \langle\phi(x)|\mathrm{d}U(x)U^\dagger(x)|\phi(x)\rangle = -\mathrm{i}\oint_{\tilde{c}} \langle\phi(x)|\mathrm{d}U(x)U^\dagger(x)|\phi(x)\rangle . \tag{57}
$$

Using equation (57) in equation (55) we obtain the final expression for $\alpha$ under the transformation $U(t)$

$$
\begin{aligned}
\alpha = \oint_{\tilde{c}} A - \mathrm{i}\oint_{\tilde{c}} \langle\phi(x)|\mathrm{d}U(x)U^\dagger(x)|\phi(x)\rangle \\
- \int_0^T \langle\tilde{\psi}_1(t)|U(t)h(t)U^\dagger(t)|\tilde{\psi}_1(t)\rangle\,\mathrm{d}t .
\end{aligned} \tag{58}
$$

The transformed geometrical and dynamical phases are given by

$$
\tilde{\beta} = \oint_{\tilde{c}} A - \mathrm{i}\oint_{\tilde{c}} \langle\phi(x)|\mathrm{d}U(x)U^\dagger(x)|\phi(x)\rangle \tag{59a}
$$

$$
\tilde{\delta} = -\int_0^T \langle\tilde{\psi}_1(t)|U(t)h(t)U^\dagger(t)|\tilde{\psi}_1(t)\rangle\,\mathrm{d}t . \tag{59b}
$$

Recall the original expression equation (15) for $\alpha$ before the transformation $U(t)$

$$
\alpha = \oint_c A - \int_0^T \langle\psi_1(t)|h(t)|\psi_1(t)\rangle\,\mathrm{d}t \tag{60}
$$

where the geometrical phase $\beta$ and the dynamical phase $\delta$ are given by equations (18) and (19). By comparing equation (59$b$) with equation (19) we see that the transformed dynamical phase is equal to the original dynamical phase

$$
\delta = \tilde{\delta} . \tag{61}
$$

This is a desirable result since the measured energy spectrum and hence the dynamical phase should be invariant under $U(t)$.

We have shown that the relative phase $\alpha$ and the dynamical phase $\delta$ are invariant under $U(t)$, this obviously implies that the geometrical phase is also invariant

$$\beta = \tilde{\beta}. \tag{62}$$

Substituting the expressions for $\beta$ and $\tilde{\beta}$ into equation (62) we see

$$\oint_c A = \oint_{\tilde{c}} A - \mathrm{i} \oint_{\tilde{c}} \langle\phi(x)|\mathrm{d}\, U(x) U^\dagger(x)|\phi(x)\rangle. \tag{63}$$

Equation (63) shows that even though the curve in projective Hilbert space has changed ($c \to \tilde{c}$), the $\mathrm{i}\langle\phi|\mathrm{d}UU^\dagger|\phi\rangle$ term compensates for this change keeping the geometric phase invariant. We can interpret $\mathrm{i}\langle\phi|\mathrm{d}UU^\dagger|\phi\rangle$ as a connection form which is induced by the transformation to a 'rotating' frame of reference. We can justify this interpretation by considering the horizontal lift equation [9]

$$\langle\chi_1(t)|\dot{\chi}_1(t)\rangle = 0 \tag{64}$$

where $|\chi_1(t)\rangle$ is the horizontal lift of the closed curve $c$ in projective Hilbert space. Equation (64) chooses horizontal tangent vectors as being orthogonal to $|\chi_1(t)\rangle$, and represents the choice of a connection. The $|\chi_1(t)\rangle$ are related to $|\psi_1(t)\rangle$ by

$$|\chi_1(t)\rangle = \exp\left[\mathrm{i}\int_0^t \langle\psi_1(t')|h(t')|\psi_1(t')\rangle\,\mathrm{d}t'\right]|\psi_1(t)\rangle. \tag{65}$$

The horizontal lift defines the geometric phase and by solving equation (64) we find

$$|\chi_1(T)\rangle = \mathrm{e}^{\mathrm{i}\beta}|\chi_1(0)\rangle \tag{66}$$

where $\beta$ is given by equation (18). Under the transformation $U(t)$ equation (64) goes into

$$\langle\tilde{\chi}_1(t)|\dot{\tilde{\chi}}_1(t)\rangle - \langle\tilde{\chi}_1(t)|\dot{U}(t)U^\dagger(t)|\tilde{\chi}_1(t)\rangle = 0 \tag{67a}$$

where $|\tilde{\chi}_1\rangle$ is the horizontal lift of the curve $\tilde{c}_1$. Recall that the curve $\tilde{c}_1$ is no longer closed. We must also consider the horizontal lift of the curve generated by the time evolution of path 2. The horizontal lift of $\tilde{c}_2$ is denoted by $|\tilde{\chi}_2\rangle$ and satisfies the equation

$$\langle\tilde{\chi}_2(t)|\dot{\tilde{\chi}}_2(t)\rangle - \langle\tilde{\chi}_2(t)|\dot{U}(t)U^\dagger(t)|\tilde{\chi}_2(t)\rangle = 0. \tag{67b}$$

The $|\tilde{\chi}_1(t)\rangle$ and $|\tilde{\chi}_2(t)\rangle$ are related to $|\tilde{\psi}_1(t)\rangle$ and $|\tilde{\psi}_2(t)\rangle$ by

$$|\tilde{\chi}_1(t)\rangle = \exp\left[\mathrm{i}\int_0^t \langle\tilde{\psi}_1(t')|U(t')h(t')U^\dagger(t')|\tilde{\psi}_1(t')\rangle\,\mathrm{d}t'\right]|\tilde{\psi}_1(t)\rangle \tag{68a}$$

$$|\tilde{\chi}_2(t)\rangle = |\tilde{\psi}_2(t)\rangle. \tag{68b}$$

We see that equations (67$a$) and (67$b$) are very similar to equations (27) and (28) upon contracting on the left with $\langle\tilde{\psi}_1|$ and $\langle\tilde{\psi}_2|$. The difference between the two sets of equations is the dynamical term $UhU^\dagger$ in equation (28$a$). The operator $UhU^\dagger$

is present only in the time evolution equation for path 1 just like the operator $h$ in equation (1). This operator represents the apparatus and gives rise to the dynamical phase. Note that the $\mathrm{i}\dot{U}U^{\dagger}$ term is common to both time evolution equations which allows it to be given a geometrical interpretation.

By expressing $|\tilde{\chi}_1\rangle$ and $|\tilde{\chi}_2\rangle$ in terms of $|\phi_1\rangle$ and $|\phi_2\rangle$, substituting these expressions into equations (67$a$) and (67$b$) and performing the same analysis that led to equation (58) we find

$$|\tilde{\chi}_1(T)\rangle = \mathrm{e}^{\mathrm{i}\tilde{\beta}}|\tilde{\chi}_2(T)\rangle \tag{69}$$

where $\tilde{\beta}$ is given by equation (59$a$). Thus, under the transformation $U(t)$ the horizontal lift equation which defines the geometric phase acquires an additional term (see equations (67$a$) and (67$b$)). This term modifies the connection and appears in the expression for the geometric phase equation (59$a$). Since the transformation $U(t)$ also effects the curve in projective Hilbert space, the line integral of the transformed connection over the transformed curve gives the same result for the geometric phase $\beta = \tilde{\beta}$.

A distinction between the energy and Hamiltonian operator has recently been made [4, 5]. The energy operator was defined so that its eigenvalues are invariant under time-dependent unitary transformations. This operator gives the measured energy spectrum and dynamical phase. The Hamiltonian transforms in a non-invariant way and should not in general be identified as the energy operator. The geometric (Berry) phase was generalized so that it was invariant under time-dependent unitary transformations by introducing a term $A_o = -\mathrm{i}U\dot{U}^{\dagger}$. These ideas are reflected in equations (59$a$) and (59$b$) where we have shown that the geometric phase remains invariant due to the additional $\mathrm{i}\mathrm{d}UU^{\dagger}$ term and that the dynamical phase is given by the operator $UhU^{\dagger}$. We have considered a time-dependent system for which the energy operator $\varepsilon(t)$ is initially equal to the Hamiltonian operator

$$\varepsilon(t) = h(t). \tag{70}$$

Under the transformation $U(t)$ the energy operator is no longer equal to the Hamiltonian. The transformed energy operator $\tilde{\varepsilon}(t)$ is just the $UhU^{\dagger}$ term in equation (28$a$)

$$\tilde{\varepsilon}(t) = U(t)\varepsilon(t)U^{\dagger}(t). \tag{71}$$

The transformed dynamical phase is given by the transformed energy operator $\tilde{\varepsilon}(t)$ (see equation (59$b$)). We have shown that the common extra term in equations (28$a$) and (28$b$) belongs to the horizontal lift equation (equations (67$a$) and (67$b$)). This term gives rise to the connection form $\mathrm{i}\langle\phi|\mathrm{d}UU^{\dagger}|\phi\rangle$ which is needed to keep the geometric phase invariant.

For periodic transformations $U(T) = U(0)$ [6], the curves $\tilde{c}_1$ and $\tilde{c}_2$ are both closed. This can be seen from equations (29$a$) and (29$b$) upon setting $t = 0$ and $t = T$ and using equations (4) and (7)

$$\tilde{\pi}_1(T) = \tilde{\pi}_1(0) \tag{72a}$$

$$\tilde{\pi}_2(T) = \tilde{\pi}_2(0). \tag{72b}$$

It is straightforward to show that the time evolution of path 2 represented by $|\tilde{\psi}_2(t)\rangle$ gives a trivial phase. Consider

$$|\tilde{\psi}_2(T)\rangle = U(T)|\psi_2(T)\rangle$$

from equation (6) we see

$$|\tilde{\psi}_2(T)\rangle = U(T)|\psi_2(0)\rangle$$

and using the periodicity of U(t)

$$|\tilde{\psi}_2(T)\rangle = |\tilde{\psi}_2(0)\rangle. \tag{73}$$

Using equations (31), (33) and (73) we can express the measured relative phase as

$$|\tilde{\psi}_1(T)\rangle = \mathrm{e}^{\mathrm{i}\alpha}|\tilde{\psi}_1(0)\rangle. \tag{74}$$

We no longer need to consider the second curve $|\tilde{\psi}_2(t)\rangle$, and can express $|\tilde{\psi}_1(t)\rangle$ in terms of a curve $|\phi_1(t)\rangle$ which is now closed ($|\phi_1(T)\rangle = |\phi_1(0)\rangle$). We can then derive equation (58) except that now the curve $\tilde{c}$ is given by $\tilde{c} = \tilde{c}_1$.

## 4. Summary

The connection in the fibre bundle: Hilbert space → space of states changes under time-dependent unitary transformations

$$A \to A - \mathrm{i}\langle\phi|\mathrm{d}UU^\dagger|\phi\rangle.$$

This is not a gauge transformation, the curvature two-form $F = dA$ also changes

$$F \to \tilde{F}$$

which implies that the geometry of the fibre bundle changes. However, the calculated physical quantities are invariant:
the relative phase

$$\alpha \to \alpha$$

the dynamical phase

$$\delta = \tilde{\delta}$$

and the geometric phase (holonomy)

$$\beta = \tilde{\beta}.$$

From a mathematical point of view the choice of a connection is in general arbitrary. It is the physical quantities which must be invariant. The connection is chosen to divide the total phase $\alpha$ into the physically observed geometrical and dynamical parts. A different choice for the connection must be made in order to calculate the observed geometrical phase under a time-dependent unitary transformation.

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